PROMYS 2025

Application Problem Set

https://www.promys.org

Welcome to the PROMYS application problem set. We have prepared 8 problems which can all be solved with no more than a standard high school mathematics background, though most of the problems require considerably more ingenuity than is usually expected in high school. Keep in mind that we are looking to see how you approach challenging problems; we do not expect you to find complete solutions to all 8 problems. Most problems will require your patience: do not rush through them. It is not unreasonable to spend a month or more thinking about the problems.

Your final writeup should include the following for each problem:

- Your final solution with clear and precise justifications.
  - If you think you have a final solution that you cannot fully justify, tell us what kind of evidence supports your belief.
  - If you cannot solve a problem, tell us about progress you have made on the problem.
- A description of your journey through the problem, including experimentations and insights that led to progress. **Do not include scratch work!**
- Any conjectures or generalizations.

For each problem, we ask that you also prepare a separate short (max 200 words) summary of your work to be typed into text boxes in the online application form. (Note: this is new for 2025.) This summary should include an overview of your work, your key insights, and justifications for your solution.

Most importantly, we ask that **you tackle these problems by yourself**. Using external resources like the internet will make it much harder for you to demonstrate your insight to us.

Here are a few suggestions for tackling these problems:

- Think carefully about the meaning of each problem.
- State and solve simpler but related problems.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.

You are welcome to use any of the various tools available for typesetting mathematics on a computer. However, we are interested in your mathematical ideas, not in your typesetting. If you prefer writing your solution by hand, we encourage you to do so. What is important is that we can read your submitted work and that you can clearly communicate your insights.

1. The squares of an infinite table are numbered as follows: in the 0-th row (i.e. the bottom-most row) and 0-th column (i.e.the left-most column) we put 0, and then in every other square we put the smallest non-negative integer that does not appear anywhere below it in the same column or anywhere to the left of it in the same row.

...	...						
6	7	...					
5	4	7	...				
4	5	6	7	...			
3	2	1	0	7	...		
2	3	0	1	6	7	...	
1	0	3	2	5	4	7	...
0	1	2	3	4	5	6	...

What number will appear in the 1607th row and 1989th column? Can you generalize?

2. The tail of a giant hare is attached by a giant rubber band to a stake in the ground. A flea is sitting on top of the stake eyeing the hare (hungrily). Seeing the flea, the hare leaps into the air and lands one kilometer from the stake (with its tail still attached to the stake by the rubber band). The flea does not give up the chase but leaps into the air and lands on the stretched rubber band one centimeter from the stake. The giant hare, seeing this, again leaps into the air and lands another kilometer from the stake (i.e., a total of two kilometers from the stake). The flea is undaunted and leaps into the air again, landing on the rubber band one centimeter further along. Once again the giant hare jumps another kilometer. The flea again leaps bravely into the air and lands another centimeter along the rubber band. If this continues indefinitely, will the flea ever catch the hare? (Assume the earth is flat and continues indefinitely in all directions.)

3. The set S contains some real numbers, according to the following three rules.

   (i) $\frac{1}{1}$ is in S.

   (ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

   (iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

   These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S? For example, by (i), $\frac{1}{1}$ is in S. By (ii), since $\frac{1}{1}$ is in S, $\frac{1}{2\cdot 1}$ is in S. Since both $\frac{1}{1}$ and $\frac{1}{2}$ are in S, (iii) tells us $\frac{1+1}{1+2}$ is in S.

4. Ten people are to be divided into 3 committees in such a way that every committee must have at least one member and no person can serve on all three committees. In how many ways can this be done?

5. Consider a 2×4 array of dots and color each dot red or blue in such a way that in each row and in each column, half the dots are red and half are blue. Now, whenever two dots of the same color are adjacent (horizontally or vertically), join them by an edge of that color. What is the difference between the number of red edges and the number of blue edges? What if we

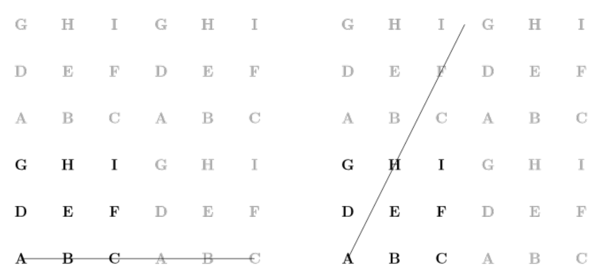
were to repeat for a 4×8 array? What if we were to repeat for an $m \times n$ array where m and n are even positive integers?

6. Consider the polynomial $(x + y + z)^n$ for $n \geq 1$. Expand and combine like-terms so that all terms of the form $x^i y^j z^k$ are together with one coefficient. Let $c_n(i, j, k)$ be this coefficient for fixed n. For any given positive integer n, can you predict how many values of $c_n(i, j, k)$ are odd? For example, for $n = 1$, $c_1(1, 0, 0)$, $c_1(0, 1, 0)$, and $c_1(0, 0, 1)$ are all odd, so there are three values of $c_n(i, j, k)$ which are odd.

7. Can you find an equilateral triangle in the xy-plane whose three vertices are within $\frac{1}{10}$ distance of a point with integer coordinates? What about within $\frac{1}{100}$?

8. Consider the grid of letters that represent points below, which is formed by repeating the bold 3×3 grid. (The points are labeled by the letters A through I.)

We'll draw lines through the bottom left point (the bold A) and at least one other bold letter, then see what set of letters the line hits. We've drawn two example lines for the repeating 3×3 grid below. For the example on the left, the set of letters is {A, B, C}, and for the right, the set of letters is {A, F, H}.



Assuming the 3×3 grid repeats forever in every direction, what is the maximum number of distinct letters a line can pass through? How many distinct sets of letters can you obtain from drawing lines in this grid?

What would happen if you had a repeated 9×9 grid of 81 distinct symbols (and still had to draw lines through the bottom left point and at least one other bold point)? What about a 6×6 grid? What about a repeated $n \times n$ grid?

End of Problem Set