PROMYS 2025

Application Problem Set

Please attempt each of the following 8 problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems.

We ask that you tackle these problems by yourself. Please see below for further information on citing any sources you use in your explorations.

Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles, or websites in your explorations, be sure to cite your sources.

Be careful if you search online for help. We are interested in your ideas, not in solutions that you have found elsewhere. If you search online for a problem and find a solution (or most of a solution), it will be much harder for you to demonstrate your insight to us.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution: this includes your experimentation and any thinking that led you to an argument.**

There are various tools available for typesetting mathematics on a computer. You are welcome to use one of these if you choose, or you are welcome to write your solutions by hand, or you might want to do a bit of both. We are interested in your mathematical ideas, not in your typesetting. What is important is that we can read all of your submitted work, and that you can include all of your ideas (including the ones that didn't fully work).

1. The squares of an infinite table are numbered as follows: in the zeroth row and zeroth column we put 0, and then in every other square we put the smallest non-negative integer that does not appear anywhere below it in the same column or anywhere to the left of it in the same row.

...	...						
6	7	...					
5	4	7	...				
4	5	6	7	...			
3	2	1	0	7	...		
2	3	0	1	6	7	...	
1	0	3	2	5	4	7	...
0	1	2	3	4	5	6	...

What number will appear in the 1607th row and 1989th column? Can you generalize?

2. The tail of a giant hare is attached by a giant rubber band to a stake in the ground. A flea is sitting on top of the stake eyeing the hare (hungrily). Seeing the flea, the hare leaps into the air and lands one kilometer from the stake (with its tail still attached to the stake by the rubber band). The flea does not give up the chase but leaps into the air and lands on the stretched rubber band one centimeter from the stake. The giant hare, seeing this, again leaps into the air and lands another kilometer from the stake (i.e., a total of two kilometers from the stake). The flea is undaunted and leaps into the air again, landing on the rubber band one centimeter further along. Once again the giant hare jumps another kilometer. The flea again leaps bravely into the air and lands another centimeter along the rubber band. If this continues indefinitely, will the flea ever catch the hare? (Assume the earth is flat and continues indefinitely in all directions.)

3. The set S contains some real numbers, according to the following three rules.

(i) $\frac{1}{1}$ is in S.

(ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

(iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S? For example, by (i), $\frac{1}{1}$ is in S. By (ii), since $\frac{1}{1}$ is in S, $\frac{1}{2\cdot 1}$ is in S. Since both $\frac{1}{1}$ and $\frac{1}{2}$ are in S, (iii) tells us $\frac{1+1}{1+2}$ is in S.

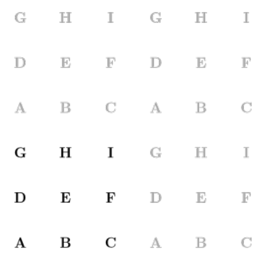
4. 10 people are to be divided into 3 committees, in such a way that every committee must have at least one member, and no person can serve on all three committees. In how many ways can this be done?

5. Consider a 2×4 array of dots and color each dot red or blue in such a way that in each row and in each column, half the dots are red and half are blue. Now, whenever two dots of the same color are adjacent (horizontally or vertically), join them by an edge of that color. What is the difference between the number of red edges and the number of blue edges? What if we were to repeat for a 4×8 array? What if we were to repeat for an $m \times n$ array where m and n are even positive integers?

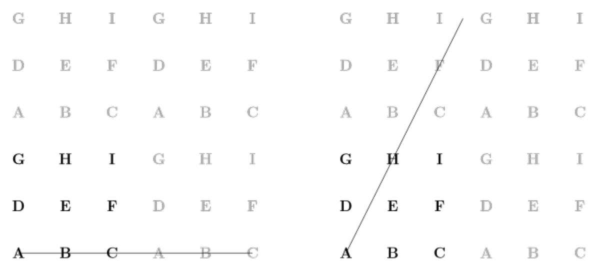
6. Consider the polynomial $(x + y + z)^n$ for $n \geq 1$. Expand and combine like-terms so that all terms of the form $x^i y^j z^k$ are gathered together with one coefficient. For any given positive integer, $n \geq 1$, can you predict how many of these coefficients are odd?

7. Can you find an equilateral triangle in the xy-plane with all its vertices having integer coordinates? If so, give an example, and if not, explain why no such triangle exists. Can you find an equilateral triangle in the xy-plane with two vertices that have integer coordinates while the third vertex is within $\frac{1}{10}$ distance of a third point with integer coordinates? What about within $\frac{1}{100}$? Within $\frac{1}{n}$?[1]

8. Consider the grid of letters that represent points below, which is formed by repeating the bold 3×3 grid. (The points are labeled by the letters A through I.)



We'll draw lines through the bottom left point (the bold A) and at least one other bold letter, then see what set of letters the line hits. We've drawn two example lines for the repeating 3×3 grid below. For the example on the left, the set of letters is {A, B, C}, and for the right, the set of letters is {A, F, H}.



Assuming the 3×3 grid repeats forever in every direction, what is the maximum number of distinct letters a line can pass through? How many distinct sets of letters can you obtain from drawing lines in this grid?

What would happen if you had a repeated 9×9 grid of 81 distinct symbols (and still had to draw lines through the bottom left point and at least one other bold point)? What about a 6×6 grid? What about a repeated $n \times n$ grid?

End of Problem Set

[1] This wording of problem 7 replaces an earlier version.