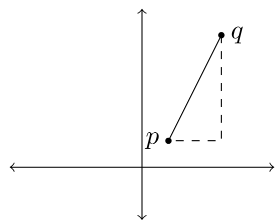
PROMYS Math Circle
Problem Set #4

These problems should help you get comfortable with the math you'll need to solve Problem of the Week #4. Enjoy!

1 Introduction to metrics

How do we measure distances in the xy-plane? Given two points p and q, we say that their distance is the length of the straight line between them.

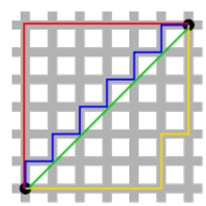


Using our usual notion of distance called the **Euclidean metric**, if p has coordinates (x_1, y_1) and q has coordinates (x_2, y_2), then we have

$$d_E(p, q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Question 1. How is the Euclidean metric formula related to the Pythagorean theorem?

This is not the only way we could measure distances, though. We could define other metrics! First, suppose we were in Manhattan, which looks like a grid.



If we want to travel from one point to another, we can't just cut through the buildings to travel in a straight line from p to q. Instead, we must travel up and then over, or over and then up. We'll call this notion of distance the **taxicab metric**, and we'll write d_T to talk about the taxicab distance between two points. We have the formula

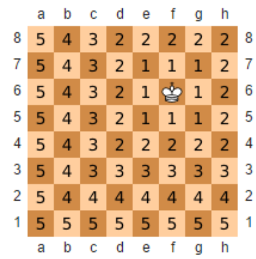
$$d_T(p, q) = |x_2 - x_1| + |y_2 - y_1|.$$

Question 2. If $p = (3, 5)$ and $q = (-4, 1)$, what is $d_T(p, q)$? Graph the two points on the plane and draw 3 different paths of length $d_T(p, q)$ which connect p and q.

A metric sphere of radius r is defined to be the set of points which are distance r from the origin (0,0). In the Euclidean metric, the a metric circle is a round circle.

Question 3. Find and then graph what the metric circle of radius 1 is for the taxicab metric.

We'll consider one other way to define the distance between two points. In chess, the king is allowed to move in any direction, up, down, left, right, or diagonally. But the king can only move one square at a time.



This picture shows how many moves it would take the king to get to different squares on the chess board. This type of metric is called the **max metric**, written d_M, because we're only concerned with the maximum of the horizontal and vertical displacements. Indeed, we have

$$d_M(p,q) = \max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

Question 4. If $p = (3,5)$ and $q = (-4,1)$, what is $d_M(p,q)$?

Question 5. Suppose you wanted the king in the picture above to move to the tile a7. Find 3 different paths of length 5 from the king to a7. (You should pass through one tile labeled 1, one tile labeled 2, etc.) Does there exist a path of length 5 from the king to a7 passing through e5? Passing through d4? Passing through c3?

Question 6. Find and then graph what the metric circle of radius 1 is for the max metric.

Now we formalize the definition of a metric. Given three points x, y, and z in space, the function d is a **metric** if it satisfies the following properties:

(i) (Positive definite) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$

(ii) (Symmetry) $d(x,y) = d(y,x)$

(iii) (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$

Question 7. Verify that the taxicab metric is positive definite. Verify that the max metric satisfies the triangle inequality.

2 Circles and π

The number π is defined to be the ratio $\frac{\text{circumference of a circle}}{\text{diameter of a circle}}$. With the Euclidean metric, we know that this ratio is always the same no matter which circle we draw, and it is approximately 3.14.

Question 8. For the taxicab and max metrics, do you think that the ratio $\frac{\text{circumference of a circle}}{\text{diameter of a circle}}$ will be the same no matter which circle we draw? Why or why not?

Question 9. Use what you found in Questions 3 and 6 to find π_T for the taxicab metric and π_M for the max metric.

3 Further thinking

Question 10. Now consider 3-dimensional space instead of the plane. The metric sphere of radius 1 is defined to be the set of points distance 1 from the origin (0,0,0). With the Euclidean metric, the collection of points distance 1 from the origin is a round sphere. What are the metric spheres of radius 1 with the taxicab and max metrics?

Question 11. In Section 2, we found the ratio π for d_T and d_M. In our normal Euclidean world, the area of a circle is $A = \pi r^2$. Using the correct ratio π for each of the other two metrics, what is the formula to find the area of their metric circles of radius r?

Question 12. Draw the triangle with vertices (0,0), (3,3), and (-3, 3). What are the lengths of the sides in d_E, d_T, and d_M? If we rotate the triangle by 45°, then it has vertices (0,0), $(0, 3\sqrt{2})$, and $(-3\sqrt{2}, 0)$. Draw this rotated triangle and determine the lengths of the sides now in each metric. For which metrics did the triangle's side lengths stay the same?

Question 13. With the Euclidean metric, an equilateral triangle has three equal angles. Find an example of an equilateral triangle in d_T and d_M such that each has a right angle.