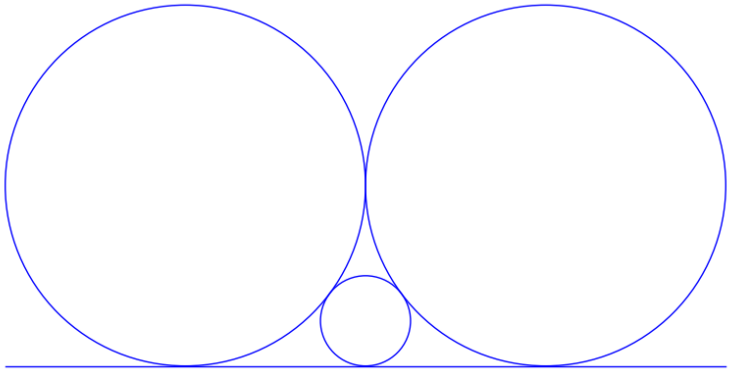
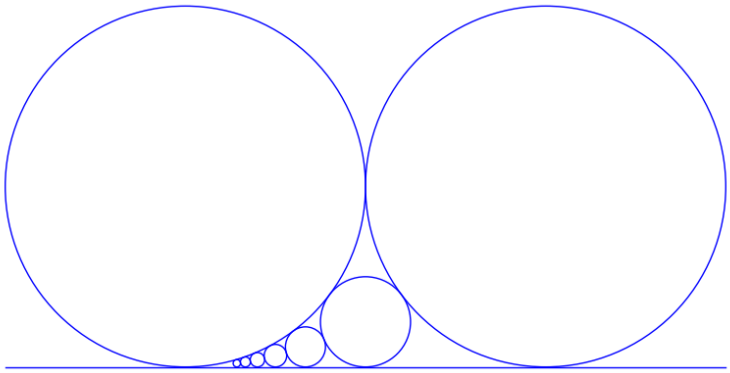
Some Geometry

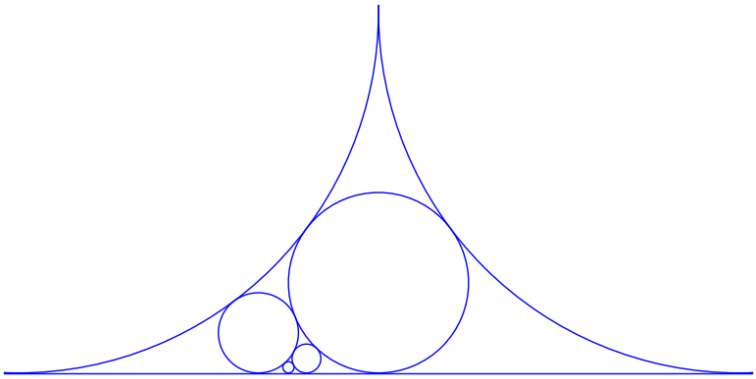
P1. In the picture below, the two large circles each have radius $\frac{1}{2}$ and are tangent to the x-axis at 0 and 1, respectively. The smaller circle is drawn so that it is tangent to both of the larger circles and to the x-axis. What is the radius of the smaller circle, and at what point is it tangent to the x-axis?



P2. You continue to draw circles in this way: choosing two circles you have already drawn, and adding the circle that is tangent to both these circles and the x-axis. If you always choose the circle on the left as one of those circles, you get the sequence of circles in the picture below. What are the radii of these circles, and at what points are they tangent to the x-axis? Any conjectures?



P3. Of course, you don't always have to choose the original circle on the left. You could choose to add circles by alternating: add one to the left of the circle you just drew, then add one to the right of that circle, etc. If you do that, you get the picture below. (So you can see more clearly, we've zoomed in on the space between the original two circles, and the circle in the middle is the small circle from the first picture.) What are the radii of these circles, and at what points are they tangent to the x-axis? Any conjectures?



P4. If you were to draw in circles infinitely, would there be a circle tangent to the x-axis at $\frac{3}{7}$? What would its radius be? What about a different rational number, $\frac{a}{b}$, where the fraction is written in lowest terms?

P5. Can you predict what the radius and point of tangency will be for a circle draw between (and tangent to) two circles, one of radius r_1 tangent to the x-axis at $\frac{a}{b}$ and the second of radius r_2 tangent to the x-axis at $\frac{c}{d}$? It may be helpful to use your prediction for how $\frac{a}{b}$ and r_1 are related.

P6. If you draw in the circles infinitely, how many circles will there be with radius $\frac{1}{8}$? With radius $\frac{1}{18}$? With radius $\frac{1}{30}$? With radius $\frac{1}{32}$? With radius $\frac{1}{72}$? Can you predict in general which radii occur, and with what frequency?

P7. Let $f(n)$ be the number of circles with radius $\frac{1}{2n^2}$ in the picture where you draw in circles infinitely (but only between the first two circles of radius $\frac{1}{2}$). What patterns do you see in the values of this function? What is $f(3)$? $f(5)$? $f(15)$?

P8. If you draw in the circles infinitely (but still only between the initial two circles of radius $\frac{1}{2}$), what can you say about the total area of all the circles? The total circumference of all the circles?