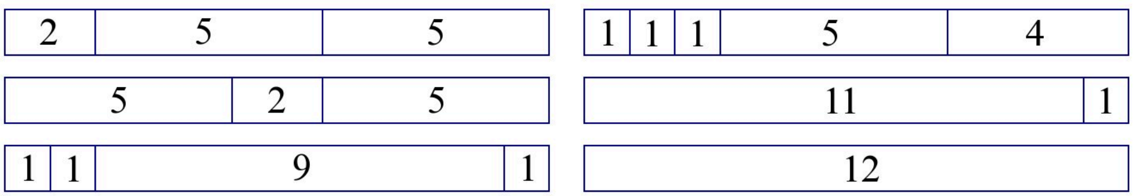
The PROMYS Math Circle

Problem of the Week #3

February 3, 2017

You can use “rods” of positive integer lengths to build “trains” that all have a common length. For instance, a “train of length 12” is a row of rods whose combined length is 12. Here are six examples:



Notice that the 2-5-5 train and the 5-2-5 train contain the same rods but are listed separately. Trains built from the same rods but in different orders are considered to be different trains.

How many trains of length 12 are there that use no rods of length 1? Can you generalize?

The PROMYS Math Circle

Stars of the Week #3

Gerardo Garcia and Maura Goncalves (City on a Hill in Dudley Square)

It was fun to read everyone's work on "Trains with no rods of length 1." (The problem statement is attached separately.) The key skill needed for solving this problem was a knack for *careful experimentation* and *keen observation*, i.e. seeing patterns and describing them.

Gerardo Garcia and Maura Goncalves of the City on a Hill School in Dudley Square have been chosen as this week's "Stars of the Week." Their 4-page write-up is attached. Gerardo and Maura worked together to look for ways of counting the number of trains of length 12 that have no rods of length 1. At first they tried to write them all down. Take a look at page four of their write-up. There you'll find a long list of trains with no rods of length 1 that was compiled by Maura. For example, she found

$$10-2,\quad 2-10,\quad 9-3,\quad 3-9,\quad 8-4,\quad 4-8\quad 8-2-2,\quad 2-8-2,\quad 8-2-2,\quad 7-5,\quad 5-7$$

and many more. She made some interesting observations that made it easier to write down examples. For example, she noticed that if you have one train of length 12, you can make more by just rearranging the rods. So when she found the $10-2$ train, she could rearrange the two cars to get the $2-10$ train. That was a good idea and it helped her write down a total of 50 different trains of length 12 with no rods of length 1 (assuming I counted right – check me!). But even with this idea, she realized she hadn't found them all and was beginning to wonder if there might be a way to know how many there are without actually writing them all down. That's exactly the kind of thing mathematicians wonder about. We're always looking for shortcuts.

At the same time that Maura was thinking of trains of length 12, Gerardo was thinking about *shorter* trains. You can see his work on page two of their write-up. He started by writing down all trains of length 2 without rods of length 1. That's easy – there's only one such train, the one with just one rod of length 2. Then he looked at trains of length 3, length 4, and lengths 5, 6, and 7. Here's a table summarizing what he found. The top row shows the length of the train, which we call n, while the second row shows the number of trains, A_n, of length n that have no rods of length 1.

n	2	3	4	5	6	7
A_n	1	1	2	3	5	8

Looking at his numbers, he seems to have somehow realized that the numbers in the second row of this table appear to follow a simple rule. Namely, the table starts with two 1's and then the next number is the sum of the first two: $2 = 1 + 1$. The next number is 3 which is the sum of the two numbers in front of it: $3 = 1 + 2$. And the next number is 5 and, sure enough, the pattern continues: $5 = 2+3$. And it works for the next number, too: $8 = 3+5$. Though it's hard to know for sure if Gerardo and Maura were surprised by this pattern, it's easy to imagine they were excited to have noticed something they didn't expect to see when they started working on this problem. They probably felt the way a scientist does when s/he is on the verge of making a new discovery. Maura and Gerardo guessed that the pattern *does* continue. It might have helped them think about the question if they had written down their conjecture using mathematical symbols, like this.

Conjecture: For all $n \geq 4$ we have $A_n = A_{n-1} + A_{n-2}$.

If this conjecture is true, then we can continue the table above. That's exactly what Maura did on page 4 of their write-up. She wrote down her table vertically, but I'll write it horizontally. Here it is:

n	2	3	4	5	6	7	8	9	10	11	12
A_n	1	1	2	3	5	8	13	21	34	55	89

If their conjecture is true, then the number of trains of length 12 with no rods of length 1 must be A_{12} which is 89 in their table.

Gerardo and Maura talked to other students and even to one of their teachers to see if others got the same results, but their numbers don't seem to have matched the numbers other people found. When that happens it can be confusing. Sometimes you just have to believe in yourself and keep going. So they started looking for a *proof* of their conjecture. That's just a fancy way of saying they were trying to figure out *why* the pattern they noticed must be true, and that means looking for deeper understanding of the problem. Based on what they've written, I think Gerardo and Maura came real close to finding a proof, but they didn't quite get it.

But I'm happy to report that they *did* get the right answer. It's true that there are exactly 89 trains of length 12 having no rod of length 1. And their conjecture is also true, and you can use it to guess how many trains of length 13 there are without rods of length 1. According to their conjecture it should be $A_{13} = 55 + 89$ which is 144. In fact, that's the right number. The conjecture always works, but *why*?

Gerardo and Maura acknowledged (on page 1 of their write-up) that they haven't proved the conjecture yet. But to be sure our answer is correct we do need to find a proof. Can you think of one? Here's an idea:

Proof of the Conjecture: For each $n \geq 4$ we want to prove $A_n = A_{n-1} + A_{n-2}$. The formula suggests that there might be a way of finding all trains of length n by first finding all trains of length $n-1$ and length $n-2$. In fact, you can do that. For each train of length $n-1$ you can make a train of length n by simply replacing the last rod with a rod that is 1 unit longer. That gives us A_{n-1} trains of length n that end with a rod of length > 2. And for each train of length $n-2$ you can make a train of length n by adding a rod of length 2 to the end. That gives us A_{n-2} trains of length n each of which ends with a rod of length 2. Altogether this gives us $A_{n-1} + A_{n-2}$ *different* trains (why are they all different?) of length n each of which has no rods of length 1 and there are no other such trains (can you see why?). So $A_n = A_{n-1} + A_{n-2}$. If you understand all of this, then you understand why the conjecture is true and the proof is complete.

Congratulations to Gerardo and Maura! Have a great weekend.

Glenn Stevens (Boston University)

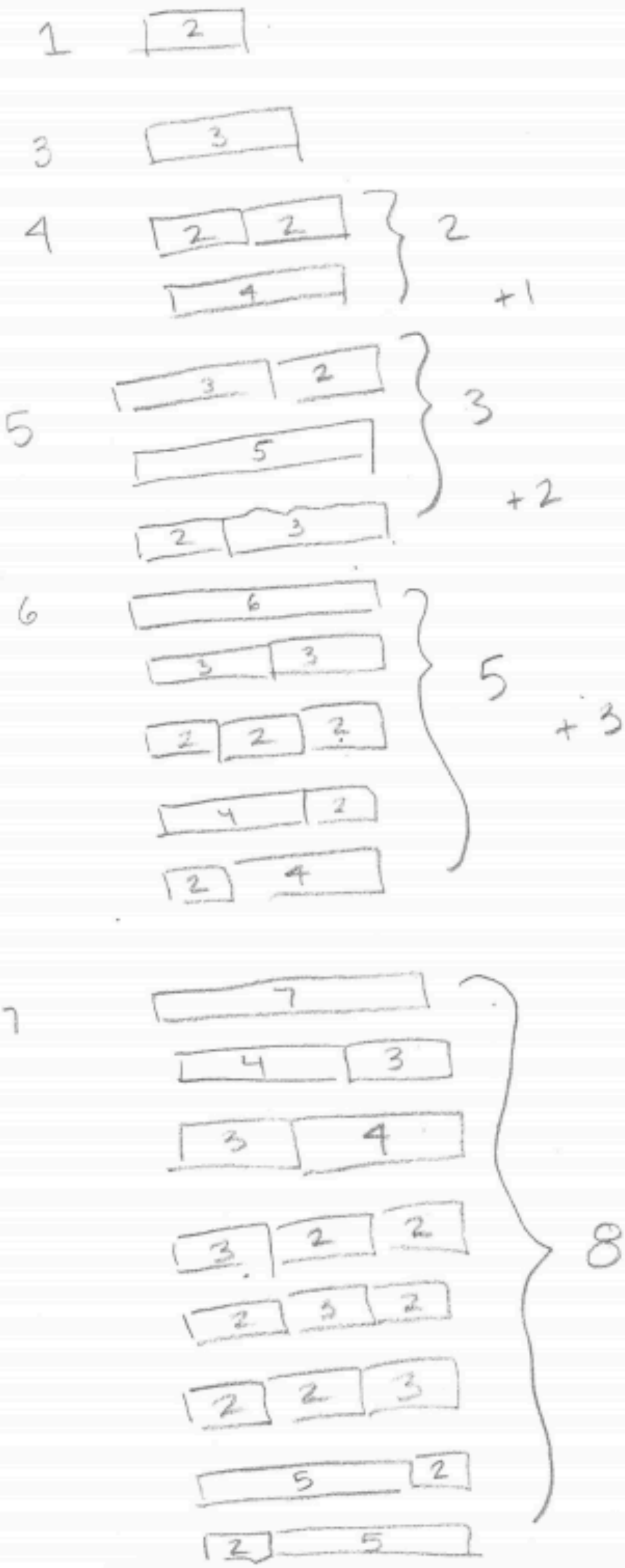
This is Gerardo Garcia’s description of his process for solving the problem.

~~I~~ ~~we~~ my partner and i started by counting with blocks to see a partern but we soon realized that took too much time so we JUSA found parents by finding out ~~to~~ how many patterns we could make from the smallest to the biggest patern.

We wrote out all of tG possible 2 unit trains, 3 unit trains ECT...

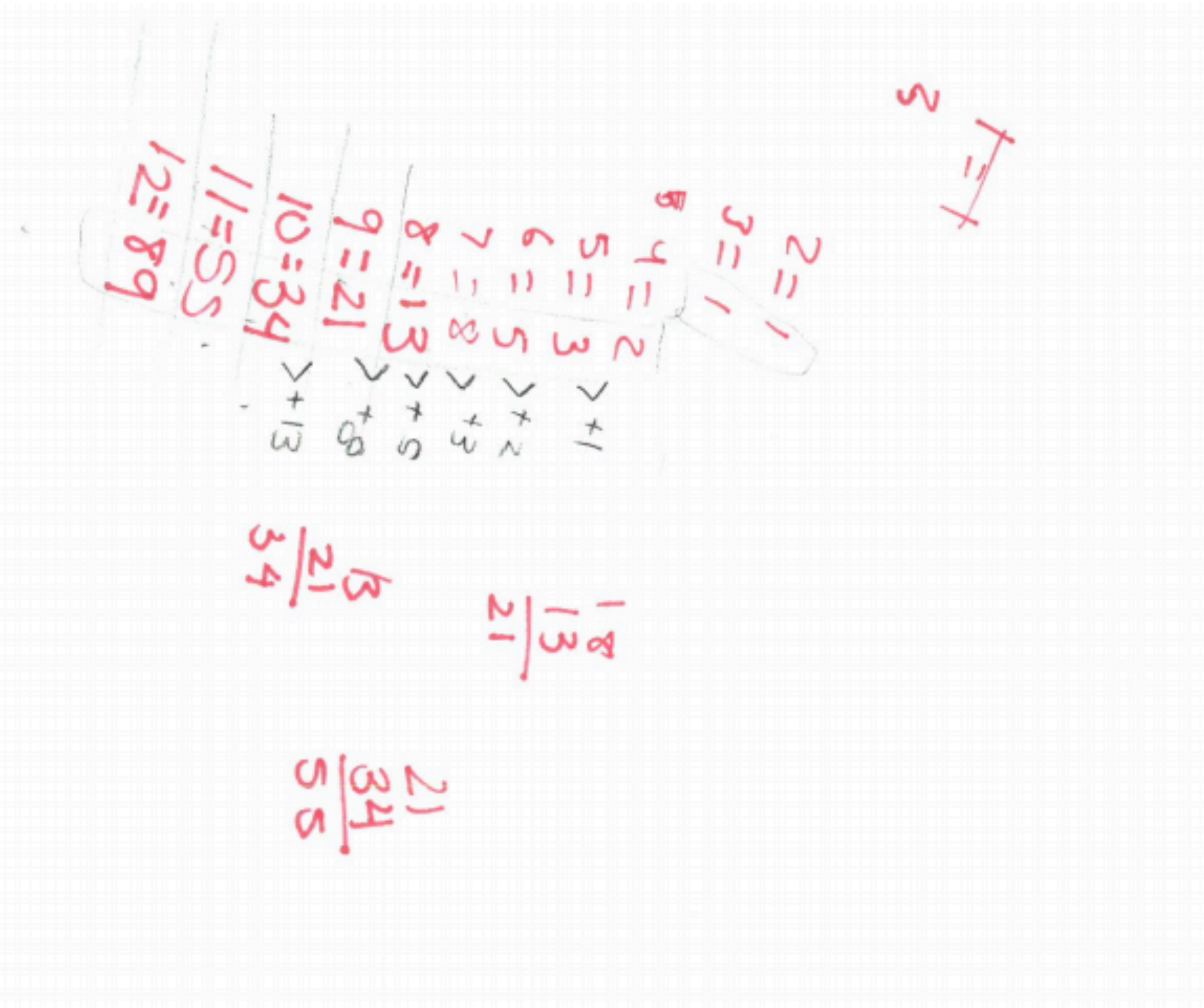
“My partner and I started by counting with blocks to see a pattern, but we soon realized that took too much time, so we just found patterns by finding out how many patterns we could make from the smallest to the biggest pattern. We wrote out all of the possible 2 unit trains, 3 unit trains, etc.”

Gerardo's work and explanation of the pattern.



The pattern is that you add the previous ~~#~~ ~~#s~~ to the next but we got a different answere from one of my teachers so we tried to proof the answer and soon reached a stand still

Maura and Gerardo's work



Maura's work. She stopped when she decided it was too much work and saw Gerardo had a pattern going—then they started working together.

students work

10-2
2-10
2!

9-3
3-9
2!

8-4
4-8
2!

8-2-2
2-8-2
2-2-8
7-5
5-7

7-2-3
7-3-2
3-7-2
3-2-7
2-3-7
2-7-3
3!

6-6
6-3-3
3-3-6
3-6-3
21

6-4-2
2-4-6
2-6-4
6-2-4
4-6-2
4-2-6
3!

6-2-2-2
2-6-2-2
2-2-6-2
2-2-2-6

5-5-2
5-2-5
2-5-5

5-3-4
4-3-5
4-5-3
5-4-3
3-4-5
3-54
3!

4-3-3-2	2-3-3-4	3-3-4-2
4-3-2-3	2-3-4-3	3-3-2-4
4-2-3-3	2-4-3-3	3-2-3-4