HABITS OF MIND: AN ORGANIZING PRINCIPLE FOR MATHEMATICS CURRICULUM

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Thinking about the future is risky business. Past experience tells us that today's first graders will graduate high school most likely facing problems that do not yet exist. Given the uncertain needs of the next generation of high school graduates, how do we decide what mathematics to teach? Should it be graph theory or solid geometry? Analytic geometry or fractal geometry? Modeling with algebra or modeling with spreadsheets?

These are the wrong questions, and designing the new curriculum around answers to them is a bad idea.

For generations, high school students have studied something in school that has been *called* mathematics, but which has very little to do with the way mathematics is created or applied outside of school. One reason for this has been a view of curriculum in which mathematics courses are seen as mechanisms for communicating established results and methods — for preparing students for life after school by giving them a bag of facts. Students learn to solve equations, find areas, and calculate interest on a loan. Given this view of mathematics, curriculum reform simply means replacing one set of established results by another one (perhaps newer or more fashionable). So, instead of studying analysis, students study discrete mathematics; instead of Euclidean geometry, they study fractal geometry; instead of probability, they learn something called data analysis. But what they do with binary trees, snowflake curves, and scatter-plots are the same things they did with hyperbolas, triangles, and binomial distributions: They learn some properties, work some problems in which they apply the properties, and move on. The contexts in which they work might be more modern, but the methods they use are just as far from mathematics as they were twenty years ago.

There is another way to think about it, and it involves turning the priorities around. Much more important than specific mathematical results are the habits of mind used by the people who create those results, and we envision a curriculum that elevates the methods by which mathematics is created, the techniques used by researchers, to a status equal to that enjoyed by the results of that research. The goal is not to train large numbers of high school students to be university mathematicians, but rather to allow high school students to become comfortable

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with ill-posed and fuzzy problems, to see the benefit of systematizing and abstraction, and to look for and develop new ways of describing situations. While it *is* necessary to infuse courses and curricula with modern content, what's even more important is to give students the tools they'll need to use, understand, and even make mathematics that doesn't yet exist.

A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics *do* and what they *say*. Such a curriculum lets students in on the process of creating, inventing, conjecturing, and experimenting; it lets them experience what goes on behind the study door *before* new results are polished and presented. It is a curriculum that encourages false starts, calculations, experiments, and special cases. Students develop the habit of reducing things to lemmas for which they have no proofs, suspending work on these lemmas and on other details until they see if assuming the lemmas will help. It helps students look for logical and heuristic connections between new ideas and old ones. A habits of mind curriculum is devoted to giving students a genuine research experience.

Of course, studying a style of work involves working on *something*, but we should construct our curricula and syllabi in a way that values how a particular piece of mathematics typifies an important research technique as much as it values the importance of the result itself. This may mean studying difference equations instead of differential equations, it may mean less emphasis on calculus and more on linear algebra, and it certainly means the inclusion of elementary number theory and combinatorics.

This view of curriculum runs far less risk of becoming obsolete before it is even implemented. Difference equations may fall out of fashion, but the algorithmic thinking behind their study certainly won't. Even if the language of linear algebra becomes less useful in the next century than it is now, the habit of using geometric language to describe algebraic phenomena (and *vice-versa*) will be a big idea decades from now. At the turn of the 20th century, the ideas and thought experiments behind number theory (the decomposition of ideals into prime factors in number fields, for example) was smiled upon as the pastime of a dedicated collection of intellectuals looking for the elusive solution to the Fermat conjecture; at the turn of the 21st century, even after it appears that Fermat is settled, these same habits of mind that led to class field theory are at the forefront of applied research in cryptography.

This approach to curriculum extends beyond mathematics, and reflection shows that certain general habits of mind cut across every discipline. There are also more mathematical habits, and finally, there are ways of thinking that are typical of specific content areas (algebra or topology, for example).

In the next sections, we describe the habits of mind we'd like students to develop. In high school, we'd like students to acquire

- some useful general habits of mind, and
- some mathematical approaches that have shown themselves worthwhile over the years.

These are general approaches. In addition, there are content-specific habits that high school graduates should have. We've concentrated on two of the several possible categories:

- some geometric habits of mind that support the mathematical approaches, and
- some algebraic ways of thinking that complement the geometric approaches.

This is a paper in progress. The first draft was for the teacher advisory board that meets once each month to give us guidance in our *Connected Geometry* curriculum development work. This current version is for a more general audience of people working in secondary mathematics education reform. A customized revision will become the introduction to the geometry curriculum we publish.

Habits of Mind

At top level, we believe that every course or academic experience in high school should be used as an opportunity to help students develop what we have come to call good general habits of mind. These general habits of mind are not the sole province of mathematics – the research historian, the house-builder, and the mechanic who correctly diagnoses what ails your car all use them. Nor are they guaranteed byproducts of learning mathematics – it is the major lament of the reform efforts that it has been shown possible for students to learn the facts and techniques that mathematicians (historians, auto diagnosticians. . .) have developed without ever understanding how mathematicians (or these others) think.

Good thinking must apparently be relearned in a variety of domains; our further remarks will be specific to the domain of mathematics. So, at top level, we'd like students to think about mathematics the way mathematicians do, and our experience tells us that they can. Of course, that doesn't mean that high school students should be able to understand the *topics* that mathematicians worry about, but it does mean that high school graduates should be accustomed to using real mathematical *methods*. They should be able to use the research techniques that have been so productive in modern mathematics, and they should be able to develop conjectures and provide supporting evidence for them. When asked to describe mathematics, they should say something like "it's about ways for solving problems" instead of "it's about triangles" or "solving equations" or "doing percent." The danger of wishing for this is that it's all too easy to turn "it's about ways for solving problems" into a curriculum that drills students in The Five Steps For Solving A Problem. That's not what we're after; we are after mental habits that allow students to develop a repertoire of general heuristics and approaches that can be applied in many different situations.

In the next pages, you'll see the word "should" a lot. Take it with a grain of salt. When we say students should do this or think like that, we mean that it would be wonderful if they did those things or thought in those ways, and that high school curricula should strive to develop these habits. We also realize full well that most students don't have these habits now, and that not everything we say they should be able to do is appropriate for every situation. We're looking to develop a repertoire of useful habits; the most important of these is the understanding of when to use what.

Students should be pattern sniffers. Criminal detection, the analysis of literature or historical events, and the understanding of personal or national psychology all require one to be on the look-out for patterns.

In the context of mathematics, we should foster within students a delight in finding hidden patterns in, for example, a table of the squares of the integers between 1 and 100. Students should be always on the look-out for short-cuts that arise from patterns in calculations (summing arithmetic series, for example). Students should fall into the habit of looking for patterns when they are given problems by someone else ("which primes are the sum of two squares?"), but the search for regularity should extend to their daily lives and should also drive the kinds of problems students pose for themselves, convincing them, for example, that there must be a test for divisibility by 7.

Students should be experimenters. Performing experiments is central in mathematical research, but experimenting is all too rare in mathematics classrooms. Simple ideas like recording results, keeping all but one variable fixed, trying very small or very large numbers, and varying parameters in regular ways are missing from the backgrounds of many high school students. When faced with a mathematical problem, a student should immediately start playing with it, using strategies that have proved successful in the past. Students should also be used to performing thought experiments, so that, without writing anything down, they can give evidence for their answers to questions like, "What kind of number do you get if you square an odd number?'

Students should also develop a healthy skepticism for experimental results. Results from empirical research can often suggest conjectures, and occasionally they can point to theoretical justifications. But mathematics is more than data-driven discovery, and students need to realize the limitations of the experimental method.

Students should be describers. Many people claim that mathematics is a language. If so, it is a superset of ordinary language that contains extra constructs and symbols, and it allows you to create, on the fly, new and expressive words and descriptions. Students should develop some expertise in playing the mathematics language game. They should be able to do things like:

- Give precise descriptions of the steps in a process. Describing what you do is an important step in understanding it. A great deal of what's called "mathematical sophistication" comes from the ability to say what you mean.
- Invent notation. One way for students to see the utility and elegance of traditional mathematical formalism is for them to struggle with the problem of describing phenomena for which ordinary language descriptions are much too cumbersome (combinatorial enumerations, for example).
- Argue. Students should be able to convince their classmates that a particular result is true or plausible by giving precise descriptions of good evidence or (even better) by showing generic calculations that actually constitute proofs.
- Write. Students should develop the habit of writing down their thoughts, results, conjectures, arguments, proofs, questions, and opinions about the mathematics they do, and they should be accustomed to polishing up these notes every now and then for presentation to others.

Formulating written and oral descriptions of your work is useful when you are part of a group of people with whom you can trade ideas. Part of students' experience should be in a classroom culture in which they work in collaboration with each other and in which they feel free to ask questions of each other and to comment on each other's work.

Students should be tinkerers. Tinkering really is at the heart of mathematical research. Students should develop the habit of taking ideas apart and putting them back together. When they do this, they should want to see what happens if something is left out or if the pieces are put back in a different way. After experimenting with a rotation followed by a translation, they should wonder what happens if you experiment with a translation followed by a rotation. When they see that every integer is the product of primes, they should wonder, for example, if every integer is the *sum* of primes. Rather than walking away from the "mistake"

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

they should ask:

- Are there *any* fractions for which this is true?
- Are there any sensible definitions for a binary operation + that would make this statement true?

Students should be inventors. Tinkering with existing machines leads to expertise at building new ones. Students should develop the habit of inventing mathematics both for utilitarian purposes and for fun. Their inventions might be rules for a game, algorithms for doing things, explanations of how things work, or even axioms for a mathematical structure.

Like most good inventions, good mathematical inventions give the impression of being innovative but not arbitrary. Even rules for a game, if the game is to intrigue anyone, must have an internal consistency and must make sense. For example, if baseball players were required, when they arrived at second base, to stop running and jump up and down five times before continuing to third, that would be arbitrary because it would not "fit" with the rest of the game, and no one would stand for it. Similarly, a Logo procedure that just produced a random squiggle on the screen wouldn't be a very interesting invention. The same could be said of those "math team" problems that ask you to investigate the properties of some silly binary operation that seems to fall out of the sky, like $\diamond$, where

$$a \diamond b = \frac{a+2b}{3}$$

It's a common misconception that mathematicians spend their time writing down arbitrary axioms and deriving consequences from them. Mathematicians *do* enjoy deriving consequences from axiom systems they invent. But the axiom systems always emerge from the experiences of the inventors; they always arise in an attempt to bring some clarity to a situation or to a collection of situations. For example, consider the following game (well, it's more than a game for some people):

> Person A offers to sell person B something for \$100. Person B offers \$50. Person A comes down to \$75, to which person B offers \$62.50. They continue haggling in this way, each time taking the average of the previous two amounts. On what amount will they converge?

This is a concrete problem, and its solution leads to a general theorem: If person A starts the game at a and person B makes an offer of b, the limit of the haggle will be $\frac{a+2b}{3}$. This might lead one to define the binary operation $\diamond$, where

$$a \diamond b = \frac{a+2b}{3}$$

and to derive some of its properties (for example, the fact that $a \diamond b$ is closer to b than it is to a explains why they never tell you how much a car costs until you make a first "offer"). The invention of $\diamond$ no longer seems arbitrary, even though the consequences of the definition might become quite playful and far removed from the original situation that motivated it.

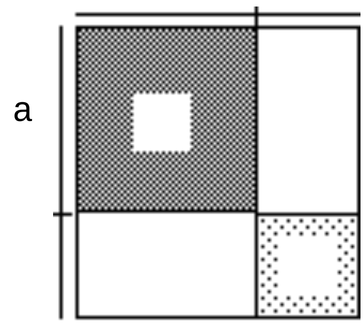
The practice of inventing a mathematical system that models a particular phenomenon is crucial to the development of mathematics.[1] Another technique mathematicians use to invent things is to take an existing system and to change one feature. That's how non-Euclidean geometry got started.

An important ingredient in the habit of inventing things is that students begin to look for *isomorphisms* between mathematical structures. It would be wonderful if students were in the habit of looking for different instances of the same mathematical structure, so that they could see, for example, that the operation of taking the union of two sets looks very much like the operation of taking the sum of two numbers.

Students should be visualizers. There are many kinds of visualization in mathematics. One involves visualizing things that are inherently visual — doing things in one's head that, in the right situation, *could* be done with one's eyes. For example, one might approach the question "How many windows are there in your house or apartment?" by constructing a mental picture and manipulating the picture in various ways.

A second involves constructing visual analogues to ideas or processes that are first encountered in non-visual realms. This includes, for example, using an area model to visualize multiplication of two binomials

[1]One reason for this is that the mathematical models often find utility outside the situations that motivated them. A classic example is the notion of a *vector space*. The notion was originally developed to describe ordinary vectors (directed line segments) in two and three dimensions, but many other mathematical objects (polynomials, matrices, and complex numbers, for example) form vector spaces.



$(a+b)^2 = a^2 + 2ab + b^2$

(or, equivalently, any two numbers like 23 and 42 or $3\frac{1}{2}$ and $8\frac{1}{3}$). The purpose of such an analogue may be to aid understanding of the process, or merely to help one keep track of a computation. Other examples include visualizations of things too small, too large, or too diverse to be seen; visualizations of *relationships* rather than objects themselves; and so on.

Finally, there are, for some people, visual accompaniments (not analogues, exactly) to totally non-visual processes. Taking the multiplication of binomials as an example again, one might actually picture the symbols moving about in some orderly fashion to help structure the computation. The imagery may not clarify *meaning* — it may just support the task, focus one's attention, or the like – but such visualizations do become part of mathematicians' repertoire.

Subdividing these three kinds of visualization a bit more finely, we get categories like these:

- Reasoning about simple subsets of plane or three dimensional space with or without the aid of drawings and pictures. This is the stuff of classical geometry, extended to include three dimensions.
- Visualizing data. Students should construct tables and graphs, and they should use these visualizations in their experiments.
- Visualizing relationships. Students should be accustomed to using the plane or space as a drawing pad to create and work with diagrams in which size is irrelevant (Venn diagrams and factor trees, for example).
- Visualizing processes. Students should think in terms of machines. All kinds of visual metaphors (meat-grinders, function machines, specialized calculators, and so on) support this kind of imagery. Students should also use *many* visual representations for the input-output pairing associated with a function, including, if the process under consideration happens to be a function from real numbers to real numbers, ordinary Cartesian graphs.
- Visualizing change. Seeing how a phenomenon varies continuously is one of the most useful habits of classical mathematics. Sometimes the phenomenon simply moves between states, as when you think of how a cylinder of fixed volume changes as you increase the radius. Other times, one thing blends into another: think of the many demonstrations that show ellipses becoming hyperbolas. This habit cuts across many of the others, including some that seem to deal with explicitly discrete phenomena.
- Visualizing calculations. There's a visual component to "mental arithmetic"

and estimation that is often ignored. Students should be in the habit of visualizing calculations (numerical and algebraic), perhaps by seeing numbers flying around in some way. A particularly useful habit in arithmetic is, given an integer, to imagine what it looks like when it is factored into primes.

Students should be conjecturers. The habit of making plausible conjectures takes time to develop, but it's central to the doing of mathematics. Students should at least be in the habit of making data-driven conjectures (about patterns in numbers, for example), but ideally, their conjectures should rest on something more than experimental evidence. For example, in predicting the behavior of the Logo procedure:

```
to inspi :side :angle :increment
 fd :side rt :angle
 inspi :side :angle+:increment :increment
end
```

students should start by experimenting with the procedure, perhaps like this:

```
inspi 5  0  1       inspi 5  5  1       inspi 5  3/10  3
inspi 5  4  2       inspi 5  3  3       inspi 5  3/2  3
inspi 5  1  3       inspi 5  2  3       inspi 5  1/2  3
```

and looking at the pictures, but their conjectures should be based on a combination of the anecdotal evidence gained from the experiments, previous experience, and *a conscious understanding and awareness of the algorithm that produces the pictures*. Here are two incorrect conjectures that one might derive from the above nine experiments:

- If $A + I = 6$, there are two pods.
- If $I = 1$ there will be two pods.

The second one is deeper than the first, and an attempt to justify it leads to an analysis of the procedure.

Students should be guessers. Guessing is a wonderful research strategy. Starting at a possible solution to a problem and working backwards (or simply checking your guesses) often helps you find a closer approximation to the desired result. Checking a guess often gets you familiar with the problem at hand; in the process of checking, students often find new insights, strategies, and approaches.

Mathematical approaches to things

The above habits of mind are quite general. There are some more specific things that are quite common in mathematics but maybe not so common outside of mathematics. Here are some of the ways mathematicians[2] approach things:

[2] Of course, by mathematicians, we mean more than just the members of AMS; we mean the people who do mathematics. Some mathematicians are children; some would never call themselves mathematicians.

Mathematicians talk big and think small. In mathematical lectures and talks, you hear things like:

> Let K be a field, V and W vector spaces over K of dimensions n and m, respectively. If $n > m$ and $T : V \to W$ is linear, then T isn't 1-1.

Mathematicians in the audience are saying to themselves:

> You can't map three dimensions into two with a matrix unless things get scrunched.

This translation to special cases is almost automatic. Of course, it requires that a collection of concrete examples is always at your fingertips. Developing this collection takes time, and it also takes a curriculum that begins with problems and examples from which general theories gradually emerge. Along these lines,

Mathematicians talk small and think big. The simplest problems and situations often turn into applications for deep mathematical theories; conversely, elaborate branches of mathematics often develop in attempts to solve problems that are quite simple to state. For example:

- Ever notice that the sum of two squares times the sum of two squares is also a sum of two squares? For example, $13 = 9 + 4$, $5 = 4 + 1$, and

  $$65 = 13 \times 5 = 16 + 49$$

  How come? A beautiful answer lies in the arithmetic of the Gaussian integers. Since $(a + bi)(a - bi) = a^2 + b^2$, our facts about 13 and 5 can be written like this:

  $$\begin{aligned} 13 &= (3 + 2i)(3 - 2i) \\ 5 &= (2 + i)(2 - i) \end{aligned}$$

  Multiply these equations together and calculate like this:

  $$\begin{aligned} 13 \times 5 &= (3 + 2i)(3 - 2i) \times (2 + i)(2 - i) \\ &= (3 + 2i)(2 + i) \times (3 - 2i)(2 - i) \\ &= (4 + 7i) \times (4 - 7i) \\ &= 16 + 49 \end{aligned}$$

- Just about all of algebraic number theory can be traced back to attempts to settle the Fermat conjecture (that there are no positive integral solutions to the equation $x^n + y^n = z^n$ if $n > 2$), a problem that no doubt came from attempts to generalize techniques for finding Pythagorean triples. The recent announcement that the conjecture has been settled (and the accompanying descriptions of what went into the proof) are perfect examples of thinking big.

Much of this "thinking big" goes under the name "abstraction." "Modeling" is also used to describe some of it. Once again, getting good at building and applying abstract theories and models comes from immersion in a motley of experiences; noticing that the sum of two squares problem connects to the Gaussian integers comes from playing with arithmetic in both the ordinary integers and in the complex numbers *and* from the habit of looking for similarities in seemingly different situations. But, experience, all by itself, doesn't do it for most students. They need explicit help in what connections to look for, in how to get started. Unfortunately (for curriculum developers), sometimes the only way to do this is to apprentice with someone who knows how to play the game.

A common technique for building abstractions and models is to derive properties of an object by studying the things you can do to it:

Mathematicians use functions. One of the effects of abstraction on mathematics is that the methods and operations of one generation become the objects of study for the next. Algebra today is the study of binary operations. Geometry after Klein and Hilbert is the study of transformations on very general geometric "objects." Sometimes, abstractions are so powerful that they can be applied to transformations on themselves; the set of mappings from one vector space to another can be given the natural structure of a vector space.

Studying the change mechanisms rather than the things that are changed is the study of functions. We've identified three broad categories of uses for functions in mathematics:

(1) **Algorithms** are useful in finding and describing coherence in calculations, in finding and describing patterns in calculations, and in finding and describing sets of repeated steps. An algorithm describes how one thing is transformed into another. It is an *algebraic* creature.
(2) **Dependences** are useful in finding and describing connections among physical phenomena (especially phenomena of physics and mechanics), in finding and describing continuous variations (especially over time), and in finding and describing causal phenomena. A dependence concentrates on how one thing is *affected* by another. It is an *analytic* creature.
(3) **Mappings** are useful in counting. A typical use of mappings is to take a well known set or structure, define a correspondence between its elements and the elements of a less well known set, and then to estimate how far away the function is from being a 1-1 correspondence between the sets. In this sense, mappings are *combinatorial* creatures.

In the materials we are developing as part of our *Connected Geometry* project, we encourage students to define functions on geometric objects (for example, the function that measures the sum of the distances from a point to the sides of a fixed triangle), use functions to solve geometric problems (find the largest box that can be made from a rectangle by cutting out little squares from the corners and folding up the sides), and help students create functions that act as translators from one point of view to another:

Mathematicians use multiple points of view. One way to look at the complex number system is through the lens of algebra (the theory of equations, for example). Another is to use analysis (continuous functions and the like). Still another is to think arithmetic (the equation $x^n - 1 = 0$), or geometry (regular polygons). But the real way to study the complex numbers is to use all these approaches at once. Many of the stunning results obtained by Gauss came from his ability to think of the same thing from several points of view (or, put another way, to equip the same set with several different structures). The book "How to Cut a Triangle" by Alexander Soifer[3] is a beautiful example of how new results come from looking at old things in unusual ways.

One very productive interaction in mathematics has been between discovery (or invention) and explanation:

Mathematicians mix deduction and experiment. There is a brewing controversy about the role of proof in mathematics curricula (especially in pre-college mathematics). Some reformers insist that students no longer need to establish their conjectures with deductive proof. This is especially true about conjectures that can be easily checked in thousands of cases with appropriate computational environments. Proof in school mathematics is seen as an add-on ritual (usually arranged in two columns) that allegedly convinces people of facts for which they need no convincing.

On the other hand, mathematics in western culture has had a 25-century love affair with proof. Ask mathematicians what makes their discipline different from others, and many will say that mathematicians prove things; they'll say that the standard for truth in mathematics is just higher than anywhere else; they'll say that mathematicians simply are not convinced of a fact, in spite of what would seem like overwhelming evidence to people in other (even scientific) disciplines, unless the fact comes with a proof.

Well, conviction comes in many ways, and truth is an elusive idea, even to people who dedicate their lives to the study of such things. The fact is that in mathematical research, proof plays very important roles that have little to do with conviction or truth. Think about the last time you worked on a problem. You probably started by experimenting, noticing something, and then wondering why. Then you said something like, "Well, it *would* be half as big if I knew that this other thing was. . . ." Right away, explanation becomes a research technique.

Proof and explanation can be used to enhance an investigation in at least two ways:

(1) *Proof establishes logical connections among statements*. When you prove a statement, you hardly ever start with first principles; instead, you establish logical *connections* between what you want and what you know. Instead of proving

> If p is a prime and p is a factor of ab, then either p is a factor of a or p is a factor of b.

you prove (if you use the typical argument to establish this result)

> If the greatest common divisor of two integers can be written as a linear combination of the two integers, then if p is a prime and p is a factor of ab, then either p is a factor of a or p is a factor of b.

[3] Center for Excellence in Mathematics Education, Colorado Springs, 1990.

It's this connection-making that is one of the most compelling reasons for looking for a proof. Making connections gives you the feeling of *explaining* your result, and that's intellectually satisfying. And, on a more pragmatic level, the connections you make often point you towards new results:

(2) *The proof of a statement suggests new theorems*. If you know that something happens on the basis of an experiment, then basically you know that the thing happens. If you are able to connect the result to something else, you have the makings of new or sharper results. For example, here's a problem we got from Wally Feurzeig (they say this actually happened at lunch one day):

> A square birthday cake is frosted on top and on the three sides. How should it be cut for 7 people if everyone is to get the same amount of cake and the same amount of frosting?

Think about this for a few minutes. Try to reflect on what you are doing as you puzzle with the problem. Perhaps you could come up with an experimental solution, but many people attack problems like this through a mixture of deduction and experiment, trying in thought experiments to picture various subdivisions and whether or not they meet the constraints stated in the problem. If you work this way and you come to a solution, the "proof" that your solution does what it's supposed to will take no work at all; it will have evolved with the construction of your particular cutting instructions. The proof isn't an "add-on" ritual that gets written after the result is established; it is an integral part of the investigation. When this happens, you can often say more than you originally intended. For example, developing the proof alongside the result lets you say: One way to cut the cake is to divide the perimeter of the cake into 7 equal parts and then to connect the subdivision points to the center of the square. In fact, this solution works for any shape cake for which there is an inscribed circle.

Mathematicians push the language. The drive to make results apply in new situations is responsible for a great deal of mathematical invention. For example, the definitions of things like 2^0 and 3^{-2} come from wanting the rules for positive integral exponents to hold in other cases. Similarly, mathematicians look for useful interpretations of negative numbered rows in Pascal's triangle, square roots of negative numbers, and so on.

Another way to say this is that mathematicians assume the existence of things they want. Suppose 2^0 existed. How would it have to behave? Sometimes, mathematicians assume the existence of things they *don't* want, hoping that they'll arrive at a contradiction. Suppose there were a polynomial with no complex root . . . . A contradiction produces the fundamental theorem of algebra.

Mathematicians use intellectual chants. A mathematician who is engrossed in a problem spends long periods of time alternating between scribbling on paper and looking off into space, kind of meditating. This second activity really involves rehashing logical connections and partial calculations, dozens (maybe hundreds) of times. There's something about taking a line of attack and repeating it to yourself over and over again that sometimes produces a breakthrough. Maybe these rehearsals of old ideas are effective because they often start to sound like familiar *other* investigations, so that subtle connections to seemingly different ideas are given a chance to surface. Building a path to this kind of mental activity into curriculum materials is tough, but it's not impossible. One way is to include short descriptions of the ruminations that occur when *we* work on problems, as Brian Harvey does in his writing for students.[4] Another way is to include interviews with reflective students who have successfully solved a problem and to ask students to reflect on and write about how they approached a problem.

Geometric approaches to things

Geometric thinking is an absolute necessity in every branch of mathematics, and, throughout history, the geometric point of view has provided exactly the right insight for many investigations (complex analysis, for example).[5] Geometers (amateurs and professionals) seem to have a special stash of tricks of the trade:

Geometers use proportional reasoning. There is a whole family of geometers (who trace their ancestry back to Euclid) for whom a real number is a ratio of two magnitudes.[6] These are the people who delight in the beautiful theorems about proportions ("the altitude to the hypotenuse is the geometric mean between the segments into which it divides the hypotenuse," for example), who are somehow able to visualize the product and quotient of two lengths, and who begin a geometric investigation by looking for constant ratios.

Visualizing proportionality is hard. Computers might help students develop proportional reasoning in a variety of ways. Measure boxes that contain ratios can show how two lengths can change size but maintain the same ratio. Software that allows one to define dilations can help students estimate scale factors necessary to map one figure onto a similar one. Proportional reasoning is a necessary ingredient in vectorial methods and in the study of fractal geometry.

Proportions in geometry often express a beautiful blending of numerical and geometric languages. This is an example of a more general phenomenon:

Geometers use several languages at once. Except in high school texts, there are no treatments of geometry that use a single technique for solving problems. Among the languages used by geometers are local languages (turtle geometry, for example), vectors (including complex numbers), "analytic" geometry (coordinates), and algebraic languages (the language of algebraic number fields). And, these languages are often used in the *same* investigation. This multiplicity of languages points to the habit of using multiple of points of view.

Even though geometric investigations are carried out with several languages, geometric *results* always sound like geometry:

[4] As in *Computer Science Logo Style* (3 vols.) MIT Press, Cambridge, MA.

[5] In fairness, algebra has also come to the rescue of geometry many times. The impossibility of the famous Greek construction problems was established only after the algebraists got involved.

[6] A "magnitude" is a length, an area, a volume, or a time span (indeed, the Greeks seem to be the first to have developed a single theory of proportions that apply to all kinds of magnitudes).

Geometers use one language for everything. For the past 150 years, the language of points, lines, angles, planes, surfaces, areas, and volumes has been applied to seemingly non-geometric phenomena, providing insight and coherence in many disparate branches of mathematics. For example, instead of saying that 1, 2, 3, and 4 are numbers that satisfy the equation $x + y + z + w = 10$, geometers (and most mathematicians), say that $(1, 2, 3, 4)$ is a "point" on the "graph" of $x + y + z + w = 10$. The entire graph of the equation (that is the collection of points that satisfy the equation) is called a "hyper-plane." Once we're calling things like $(1, 2, 3, 4)$ points, we might as well talk about vectors, and then we can define orthogonal vectors, and even the angle between two vectors.

The strategy is to take a familiar geometric idea, say the cosine of the angle between two vectors A and B, find a description of that idea that makes sense for the generalization (in this case, some algebraic expression[7] that gives the cosine), show that the description can be used as a *definition* of the idea for generalized "vectors" (in this case, you'd want the algebraic expression to always take values between -1 and 1, for example), and then to work with this new definition using familiar geometric language.

In one sense, this is a game, an example of extending the language. But it's more than a game: By finding a way to use the language of geometry to describe a new situation, we get a whole collection of insights that might be true in the new domain. So, in our geometry of points that look like $(1, 2, 3, 4)$, what's the proper analogy for a triangle? Do the angles of a triangle add up to 180°? Do two planes intersect in a "line?" Questions like these often point up fruitful lines of investigation. They also make geometry more powerful, because they extend the domain over which geometric facts apply.

An example of using geometry-talk to gain new ways to look at things is in number theory. Around the turn of the century (this one), the mathematician Hensel was investigating ways for solving algebraic congruences modulo powers of primes. He invented a collection of techniques that he turned into a number system, the p-adic integers (p is a prime), and as the work progressed, it began to borrow heavily from the language of geometry. The geometry in the p-adic integers was strange indeed: every triangle was isosceles, and circles had infinitely many centers. But, once you get used to this strange land, geometric language gives you some ideas about what to expect, and it provides you with some interesting slants on arithmetic. As it turns out, the geometric analogies were more than just analogies: It's possible to realize the geometry of the p-adic integers as the geometry of very simple fractal-like subsets of the plane. So, things come full circle: The language of geometry is transported to a non-geometric situation as an aid to describing arithmetic phenomena. But then the language suggests that there might actually *be* an underlying geometry after all, and it turns out that the "non-geometric" situation has a concrete geometric model.

Now, all mathematicians appreciate the way that geometric language gives coherence to their discipline, but geometers seem to like another aspect of this approach:

[7] The cosine of the angle between A and B in two or three dimensions is given by $\frac{A \cdot B}{\|A\| \, \|B\|}$, where the numerator is the dot product and the denominator is the product of two lengths (which can be expressed as dot products, too).

they love to use words like point and line when they are really talking about, say, numbers and sets, because they love the way everything hangs together:

Geometers love systems. We know a teacher in Ohio, Bill Kramer, who has been enjoying a hobby for about 20 years: Bill has defined a geometry that contains 25 points.[8] He has defined lines, triangles, measures, even rotations on his 25 points, and his hobby consists of seeing how far he can push the analogy with Euclidean geometry in this finite world. What attracts Bill to this work is the logical connectedness of it all; he asks what a reasonable definition for, say parallel lines would be, and then he sees if the classical theorems about parallels and, say, angles, hold up in his system.

Geometers like another kind of systematizing in which many special cases are combined into one large result. One way to do this is to look at *families* of geometric events:

Geometers worry about things that change. Because geometry was originally developed to describe two and three dimensional space, reasoning by continuity has always had an attraction for geometers. Continuity can be used to systematize disparate results. So, an angle formed by two chords has measure equal to half the sum of its arcs. Move the vertex of the angle towards the circle; one arc goes to 0 and the angle becomes an inscribed angle, and a new theorem is born. Then move the vertex outside the circle, to get another result, and finally, if you like, move it to infinity to see that parallel chords subtend equal arcs.

Dynamic geometry (*The Geometer's Sketchpad* or *Cabri*, for example) software can support students in their development of this habit of mind. At the very least, it can be used to develop conjectures. Think, for example, of a segment parallel to the bases of a trapezoid and connecting the non-parallel sides. Its length varies continuously between the longest base and the shortest one. Somewhere, it should be the average of the two. Where?

Sometimes, you expect things to change and they don't. Eventually, you learn how *useful* that is:

Geometers worry about things that don't change. Suppose you take a small rotation of, say, 2°, followed by a big vertical translation, say 80 feet straight up. What is the resulting map? A little experimenting suggests that it might be a rotation about some distant point. How could you check things further? One way would be to try to find the center of the alleged rotation. And one way to look for this center is to look for a point that *doesn't move* under the transformation.

This searching for invariants under transformations is a key ingredient in geometric investigations. For certain kinds of maps, this leads you into the theory of eigenvalues. For other kinds, you start thinking about topological invariants. Klein distinguished different *geometries* by the *theorems* that stayed true under the action of the respective transformation groups.

The habit of looking for invariants comes into play in another context: Invariance can be used to show that a given construction produces a well-defined function. The theorem about the "power of a point" is one of these: Define a function on $\mathbb{R}^2$ by drawing a line from a point P that intersects a circle O in two points A and B (A

[8] REF???

and B might be the same). Then the value of the function at P is defined as the product $PA \times PB$. The theorem is that this function is well-defined: It doesn't matter what line you draw through P.

One last thing about geometers:

Geometers love shapes. There is absolutely nothing to say here beyond what Marjorie Senechal says in her beautiful piece *Shape* in "On the Shoulders of Giants." In that article, Senechal breaks the study of shape into four broad categories. In addition to visualization, these include:

- Classification. Geometers classify shapes by congruence and similarity, by combinatorial properties (numbers of vertices or edges, for example), and by topological properties (number of "holes," for example).
- Analysis. Tools used to analyze shapes include symmetry (including self-similarity), regularity (tiling and packing properties), dissection, and combinatorics.
- Representation. Representations include models, drawings, computer graphics, maps, and projections.

Just look in a book or paper written by a geometer (Senechal or Coxeter, for example). There are pictures.

Algebraic approaches to things

In late 1993, the U.S. Department of Education Office of Research sponsored a colloquium as a first step in a major effort, the Algebra Initiative, that will rethink the importance of algebra and algebraic thinking from kindergarten through graduate school. The charge for the colloquium begins with motto: "Algebra is the language of mathematics."

Algebra is *a* language for expressing mathematical ideas (there are certainly others), and, like any language, it consists of much more than a way to represent objects with symbols. There are algebraic habits of mind that center around ways to transform the symbols. For algebraists, the images of these transformations are so strong and pervasive that the symbols take on a life of their own, until they become objects that exist as tools for informing one about the nature of the transformations.

People who are in "algebra mode" use a special collection of habits of mind:

Algebraists like a good calculation. Underneath it all, algebra is the study of sets equipped with one or more binary operations. The spirit of algebra is the study of how to reason about the behavior of these binary operations. A set equipped with binary operations is a system in which one can calculate, and algebra asks the question, "What are the rules for calculating in this system?" The calculations can be with numbers, abstract symbols, functions, propositions, permutations, even calculations. Sometimes the calculations are just for fun, as in the famous:

$$\left(\frac{\dfrac{1}{x^2} + \dfrac{1}{y^2}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}} - \frac{\dfrac{1}{x^2} - \dfrac{1}{y^2}}{\dfrac{1}{x^2} + \dfrac{1}{y^2}} \right) \div \left(\frac{8}{\left(\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y}\right)\left(\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} - 2\right)} \right)$$

But most often the calculations are for a purpose.

For example, Marvin Freedman of Boston University tells the story of being intrigued as a child with a puzzle that led to a card trick:

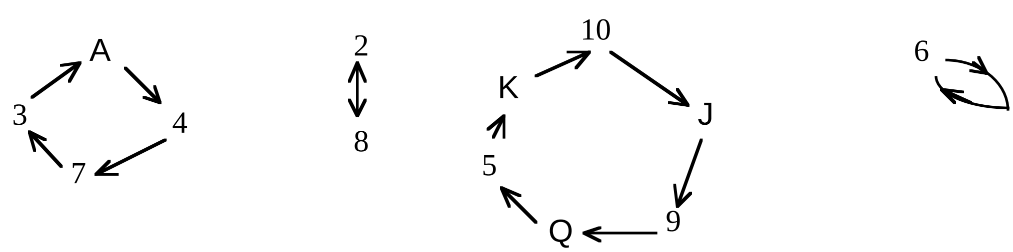
> Take a deck of 13 cards in the same suit. Perform the following shuffle to the deck: For each letter in the word "ONE," move a card from the top of the deck to the bottom (this moves 3 cards to the bottom). Turn over the new top card and put it on the table. Then spell out the word "TWO," and, with each letter, move a card from the top to the bottom of the deck (3 more cards go to the bottom). Put the new top card face up on top of the first card on the table. Then spell out the word "THREE," and, with each letter, move a card from the top to the bottom of the deck (5 cards go to the bottom). Put the new top card face up on top of the pile on the table. Keep doing this until you can't anymore. What initial arrangement of the 13 cards will cause the pile on the table to be in natural order (ace through King) when it is picked up and turned over?

Marvin discovered that if he *started* with the cards in natural order, and if he kept performing the shuffle to the pile on the table, then after 12 iterations, the pile on the table was in natural order. That meant that if he arranged the cards by performing the operation 11 times, then one more shuffle would put things right.

It took a few years before Marvin could understand the reason behind his trick; in middle school, armed with a sense for algebraic thinking, he represented the shuffle by a permutation T:

$$T = \begin{pmatrix} A & 2 & 3 & \dots & Q & K \\ 4 & 8 & A & \dots & 5 & 10 \end{pmatrix}$$

and he drew a cycle graph:



This allowed him to conclude that $T^{12} = 1$, (where 1 stands for the identity permutation), so $T^{11}\,T = 1$, showing that if T is performed to the image of T^{11}, you'll get the cards in order.

Of course, Marvin's method is completely general in the sense that it can be used to find the right pre-arrangement of cards for *any* shuffle. It's about calculations, not with numbers, but with objects invented for a specific purpose. But these card-shuffle-like objects can be used in other situations (to describe the rigid motions of a cube, for example), and, for a given deck-size, they can be gathered up into a set, composed, decomposed, inverted, and transformed. They thus take on an existence of their own, forming a little (well, not so little – for a deck of 13 cards, there are 13! of them) system (or *structure*) in which you can perform and reason about calculations. The permutations become objects of study in their own right and the calculations become calculations with permutations rather than with decks of cards. This is a perfect example of the next habit:

Algebraists use abstraction. The word "abstract" has taken on negative connotations in the mathematics education community, where it is often used as an opposite to "concrete" or even "simple" or "clear." In algebra,[9] abstraction is a natural and powerful tool for expressing ideas and obtaining new insights and results.

Because algebra is so tied up with calculations, the habit of abstraction in algebra is often activated when an algebraist finds two systems calculating the same. For example, the ordinary integers have many of the same arithmetic properties as the arithmetic of polynomials (in one variable with rational coefficients). In both systems, you can factor things into primes, you can perform division with remainder, and you can find greatest common divisors. What's more important is that the *algorithms* for calculating these things are almost identical. That leads an algebraist to invent a structure that captures the similarities.

Sometimes, it's possible to abstract off some features of a situation on the basis of *one* example. Suppose you were studying the behavior of the roots of $x^5 - 1 = 0$. One root of this equation is 1, and the fundamental theorem of algebra (along with the factor theorem) implies that there are four other ones in the complex numbers.[10] Suppose ζ is one of these. Then $\zeta^5 = 1$. Consider the complex number ζ^2. Since $(\zeta^2)^5 = (\zeta^5)^2 = 1^2 = 1$, ζ^2 is another root of the equation. In fact, this argument shows that *any* power of ζ is a root of the equation (because $(\zeta^n)^5 = (\zeta^5)^n = 1^n = 1$). How can this be? The equation $x^5 - 1 = 0$ has at most 5 roots, and it looks like we've produced infinitely many. Some powers of ζ must be the same. Well, it turns out that the roots $1 = \zeta^0$, ζ, ζ^2, ζ^3, and ζ^4 are all different,[11] and there are 5 of them, so these first five powers are *all* of them. This means that any other power, ζ^{247}, for example, must be one of the five numbers 1, ζ, ζ^2, ζ^3, and ζ^4 Which one? Well, since $\zeta^5 = 1$,

$$\begin{aligned}
\zeta^{247} &= \zeta^{245+2} \\
&= \zeta^{245} \times \zeta^2 \\
&= \zeta^{5\times 49} \times \zeta^2 \\
&= \left(\zeta^5\right)^{49} \times \zeta^2 \\
&= 1^{49} \times \zeta^2 \\
&= \zeta^2
\end{aligned}$$

[9] and in many other parts of mathematics

[10] In fact, Demoivre's theorem allows you to write them down.

[11] If, for example, $\zeta^4 = \zeta^3$, then $\zeta^3(\zeta - 1) = 0$, so either $\zeta = 0$ (which it isn't, because the fifth power of 0 isn't 1) or $\zeta = 1$ (which it isn't, because we picked it to be different from 1).

so $\zeta^{247} = \zeta^2$ To find any power of ζ, then, you can "ignore" multiples of 5 in the exponent. So, we can confine our attention to the set

$$\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$$

Notice that if we multiply two elements from our set

$$\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$$

the product must be in the set (because the product of two powers of ζ is another power of ζ, hence a root of $x^5 - 1 = 0$, and our set consists of all such roots). For example:

$$\begin{aligned} \zeta^2 \cdot \zeta^4 &= \zeta^6 = \zeta^1, \\ \zeta^3 \cdot \zeta^4 &= \zeta^7 = \zeta^2, \quad \text{and} \\ \zeta^4 \cdot \zeta^4 &= \zeta^8 = \zeta^3 \end{aligned}$$

We now have a little "system" in which we can calculate. In fact, calculating simply involves calculating with the exponents. And calculating with the exponents is especially simple: to multiply two powers of ζ together, add the exponents, divide this sum by 5 and take the remainder, and raise ζ to this power. Indeed, instead of working with the actual powers of ζ, we can calculate with the five exponents:

$$\{0, 1, 2, 3, 4\}$$

The binary operation on these exponents isn't usual addition or multiplication, it's a new thing, call it $\oplus$, where:

$$a \oplus b = \text{the remainder you get when } a + b \text{ is divided by } 5$$

We are now working in an abstract system, and we can forget the fact that the elements of our system stand for exponents of fifth roots of 1. We can calculate away, making conjectures and verifying them (for example, every element has an inverse under $\oplus$), we can add new features to our system (another binary operation, say), and, if this abstraction proves worthwhile, apply our calculations to situations quite remote from roots of unity. Of course, in this case it *does* prove worthwhile: we have built the additive group in $\mathbb{Z}/5\mathbb{Z}$, an an object that comes up throughout algebra.

Algebraists like algorithms. Algebra began as a search for algorithms for solving equations, and algebra has never lost its taste for finding recipes for solving classes of problems. Algebraic algorithms come in all sorts. Some provide shortcuts for calculations that could, in principle, be carried out. Others tell you about properties of algebraic objects that would be quite difficult to determine without the algorithms. Most have the characteristic that, if you're not in on the process of designing them, they seem quite astounding. On the other hand, for the designer of an algorithm, the finished product is often the result of capturing the essence of extensive calculations. Here are some examples of how algorithms are used in algebra and how algebraic algorithms are applied outside algebra:

- Sometimes algorithms are quite simple: to find a polynomial whose roots are the reciprocals of the roots of a given polynomial, write the coefficients in reverse order. So, $3x^4 - 2x^2 + 5x + 6$ and $6x^4 + 5x^3 - 2x^2 + 3$ have reciprocal roots.
- Many algorithms are *inductively defined*; the rule that describes them outlines a recursive process. For example, if a and b are integers, there are integers x and y so that $xa + yb = \gcd(a, b)$. How can you find x and y? Well, x and y are clearly functions of a and b, so let's call them $x(a, b)$ and $y(a, b)$. Then these equations outline an algorithm for calculating their values:

$$x(a,b) = \begin{cases} 0 & \text{if } a = 0 \\ y(b \bmod a, a) - \lfloor \frac{b}{a} \rfloor \cdot x(b \bmod a, a) & \text{otherwise} \end{cases}$$

and

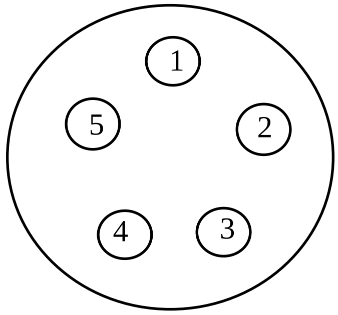
$$y(a,b) = \begin{cases} 1 & \text{if } a = 0 \\ x(b \bmod a, a) & \text{otherwise} \end{cases}$$

(Here, $\lfloor \frac{b}{a} \rfloor$ means the integer quotient that you get when b is divided by a.) Trying this our for particular a and b (say, $a = 124$ and $b = 1028$) shows that these equations expand into a significant calculation.
- Given two polynomials, say $f = 3x^3 + 5x^2 - 3x + 1$ and $g = x^2 + 5x - 6$, here's an algorithm for determining if they share a common root: Create a matrix from the coefficients of f and g by writing down the coefficients of f twice (because the degree of g is 2) and writing down the coefficients of g three times (because the degree of f is 3), using the following pattern:

$$\begin{pmatrix} 3 & 5 & -3 & 1 & 0 \\ 0 & 3 & 5 & -3 & 1 \\ 1 & 5 & -6 & 0 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 0 & 1 & 5 & -6 \end{pmatrix}$$

Then the polynomials share a root if and only if the determinant of this matrix is 0.
- Algorithms that involve algebraic calculations often apply outside algebra; in many parts of mathematics, the algebra of ordinary polynomials can be used as a technique for keeping track of information.

For example, a Simplex lock is a combination lock that shows up in dormitories, hotels, and airports:



A combination consists of an ordered set of "pushes," and each push is a collection of buttons (one or more) that are pushed together. For example, one combination might be "push 3 and 5 together, then push 1, 2, and 4 together" or "push 2, then push 3 and 4 together." How many combinations are there? What if the lock had n buttons? Here's an algorithm for finding out:

Consider the function ψ, a transformation defined on polynomials in one variable x with, say, integer coefficients, according to the rule:

$$\psi(f(x)) = xf(x) + (x+1)f(x+1)$$

So, for example,

$$\begin{aligned}\psi(2x+1) &= x(2x+1) + (x+1)(2(x+1)+1) \\ &= 4x^2 + 6x + 3\end{aligned}$$

The algorithm is simply

> To find the number of combinations on an n-button lock, iterate ψ n times, starting with the constant polynomial 1 and double the constant term.

So, we can calculate like this:

$$\begin{aligned}\psi(1) &= x \cdot 1 + (x+1) \cdot 1 = 1 + 2x \\ \psi(1+2x) &= 3 + 6x + 4x^2 \\ \psi(3 + 6x + 4x^2) &= 13 + 30x + 24x^2 + 8x^3 \\ \psi(13 + 30x + 24x^2 + 8x^3) &= 75 + 190x + 180x^2 + 80^3x + 16x^4 \\ \psi(75 + 190x + 180x^2 + 80^3x + 16x^4) &= 541 + 1470x + 1560x^2 + 840x^3 + 240x^4 + 32x^5\end{aligned}$$

and so on. So, a lock with 2 buttons has 6 combinations (including the "empty" combination), a lock with 3 buttons has 26, a four-button lock has 150 combinations, and a five-button lock (the one Simplex sells) has 1082.

Algebraists break things into parts. A useful technique in algebra is to identify the "building blocks" of a structure. Algebraists like "structure theorems" (or "decomposition theorems") that usually say something like "every object under consideration is a combination of a collection of very simple objects." The most famous decomposition theorem is the fundamental theorem of arithmetic: every integer except for 0, 1, and -1 can written (in essentially one way) as a product of primes. Hence, with respect to the operation of multiplication, primes are the building blocks for integers. The reason that structure theorems are so desirable is that results about the building blocks can usually be extended to results about more general objects. For example, knowing that, for a prime p and a non-negative integer e, p^e has $e+1$ positive factors yields a simple algorithm to find the number of divisors for any positive integer.

Algebraists often devise decomposition theorems for classes of algebraic *structures* as well as for more atomic things (like integers). So, linear algebra is full of ways to decompose a vector space into useful subspaces, and a basic result in group theory shows how to decompose any finite commutative group into cyclic groups.

Another decomposition technique in algebra is to break a structure up into classes with respect to some equivalence relation. In many cases, this is just an abstraction mechanism for expressing similarities among various elements of a structure. For example, when we were looking at how integers behave when they are exponents for a fifth root of 1, we saw that two integers behave the same if they differ by a multiple of 5. So, with respect to the situation at hand, the integers break up into 5 classes, each class containing all the integers that leave the same remainder when they are divided by 5. This has the effect of equating all multiples of 5 to 0, all numbers of the form $5k+1$ to 1, and so on.[12]

Algebraists extend things. The calculations, algorithms, and decompositions described above all take place in algebraic systems (sets of things that are equipped with binary operations that allow you to calculate). New insights come when you see how a calculation or theorem behaves when you put a given system inside a larger one.

For example, the ordinary integers sit inside the Gaussian integers. How does arithmetic change when you move from ordinary integers to Gaussian integers? Well, there is still a fundamental theorem of arithmetic, but the collection of primes is different. Some primes in the ordinary integers stay prime in the larger system (3, for example), and some do not ($5 = (2+i)(2-i)$). Right away, a question emerges: Which integer primes stay prime in the Gaussian integers and which primes don't? It turns out that you can tell quite simply, using a test that only involves arithmetic with ordinary integers: an odd prime stays prime if it leaves a remainder of 3 when divided by 4, and an odd prime splits into two (Gaussian integer) prime factors if it leaves a remainder of 1 when divided by 4. The integer 2 factors is a special way: it is essentially the square of $1+i$: $2 = -i(1+i)^2$. The fact that the behavior of a prime integer in an extension of the ordinary integers is determined by information that is already *in* the ordinary integers (in the example described here, how the prime behaves with respect to 4) is a special case of one of the central theorems

[12]In many situations, algebraists use this mechanism to "get rid of" a troublesome element: If 7 is causing you trouble, work in the integers mod 7 instead of the ordinary integers.

in a major branch of algebra (class field theory) that developed in the twentieth century.

The habit of extending the system under consideration is used all over algebra. Modern interpretations of the work of Galois in the theory of equations depend heavily on the extension idea. Many of the flawed proofs of the Fermat conjecture that have emerged over the years make a mistake in one way or another of assuming that certain properties (like the fundamental theorem of arithmetic) remain true under extension. In pre-college algebra, students are asked several times to enlarge the number systems in which they calculate, starting with the natural numbers in elementary school and ending with the complex numbers in high school.

Algebraists represent things. There are formal mathematical definitions of representations, in which the elements of one algebraic sturcture correspond to certain functions on another, but we adopt a broader and more informal use of the word here. Essentially, the idea is to use a well-understood structure to study a less well known one or to set up an interplay between seemingly different structures that proves fruitful in the study of both. Linear Algebra abounds with examples of such representations. One of the most important for beginning students often goes unmentioned in many courses: the representing of points on the plane (or in space) with ordered pairs (triples) of real numbers. These bijections between the plane and $\mathbb{R}^2$ and between three dimensional space and $\mathbb{R}^3$ are two of the most profound in all of mathematics. Their study is begun in analytic geometry. The contribution of Linear Algebra is to equip $\mathbb{R}^2$ and $\mathbb{R}^3$ with the structures of vector spaces (so that the elements can be added and scaled), giving an algebraic perspective to ordinary Euclidean geometry.

There are many other examples of representations in algebra. The representation of linear transformations on Euclidean space by matrices (so that the sum and composition of the transformations correspond the the sum and matrix product of the associated transformations) is one of the biggest contributions linear algebra has made to modern mathematics. In group theory, for example, mathematicians from Frobenius to Gorenstein have used matrix representations of finite groups as a basic research technique.

Why habits of mind?

The mathematics developed in this century will be the basis for the technological and scientific innovations developed in the next one. The thought processes, the ways of looking at things, the habits of mind used by mathematicians, computer scientists, and scientists will be mirrored in systems that will influence almost every aspect of our daily lives.

If we really want to empower our students for life after school, we need to prepare them to be able to use, understand, control, and modify a class of technology that doesn't yet exist. That means we have to help them develop genuinely mathematical ways of thinking. In this paper, we've tried to describe some of these mental habits. Our curriculum development efforts will attempt to provide students with the kinds of experiences that will help develop these habits and put them into practice.

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