**The PROMYS Math Circle**

Problem of the Week #5

Send solutions to Demi Dubois (demid@bu.edu) by Friday, February 17, 2017

Starting with the numbers 1 2 3 4, you can put + and - signs between them and compute the result in many ways. For example, by using all + signs you get 10: $1+2+3+4=10$. By making a different choice you can also get 0. Here's how:

$$1-2-3+4=0.$$

Is it possible to put + and − signs between the numbers 1 2 3 ... 50 in some way so that the result comes out to 0? What about the numbers 1 2 3 ... 100?

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# Star of Week #5

## Huanheng Wu

(Brighton High School)

It was interesting to read everyone's responses to this problem. Most of you seemed to realize instinctively that the answer had something to do with whether or not the sum of the numbers from 1 to 50 (or 1 to 100) is even or odd. Many of you said that since

$$1 + 2 + 3 + \ldots + 50 = 1275$$

is an *odd* integer, it's not possible to switch the signs of some of those numbers in order to make the sum add up to 0. But no one really said clearly *why* it's not possible. And many of you also said that since

$$1 + 2 + 3 + \ldots + 100 = 5050$$

is an *even* integer, it *is* possible to switch some of the signs around to make the sum add up to 0. Then many of you went on to find particular ways of doing it. In both of these cases, your instincts were absolutely correct. But knowing the answer and being able to explain *why* your answer is correct are different things entirely.

The Star of Week #5 is Huanheng Wu of Brighton High School. Take a look at the work he submitted to us. You can see his scratch work as well as some of his more finished ideas. Let's begin with the third page of his notes. You'll see there that he does something very important – *he tries a few simple cases before attempting to solve the general case.* This is always a good idea when you're trying to figure out what's going on. In fact, he sees some interesting stuff in these simple cases – it's worth looking at. He starts with the numbers 1, 2, 3, 4, and tries a few ways of arranging the $\pm$ signs. He sees right away that you can get a sum of zero by putting a minus sign in front of the 2 and in front of the 3 and then adding up:

$$1 - 2 - 3 + 4 = 0.$$

Interesting that he added the remark "$4/2 = 2$ and 2 is an even number." Oddly enough he doesn't tell us why that comment is important to him, *but keep reading!* Then he tries to do the same thing with the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. He writes something like this:

$$1 - 2 + 3 - 4 + [+5 + 6] - 7 + 8 - 9 + 10.$$

I put the brackets around the $+5 + 6$ because we can see from his erasures that he wasn't sure what to do at this point. But then he notices that if you add all the numbers except the 5 and the 6, the sum is zero. I wonder what he was thinking at that point – we can only guess, but probably he noticed that if you add *all* the numbers you get 11 and there's no obvious way (maybe there's no way at all?) to change the signs around to get the sum down to 0.

So next he tried the numbers 1 to 12 and remarks that $12/2 = 6$ and 6 is even (*why did he say that?*) and then he draws a very beautiful row of numbers:

$$1 - 2 + 3 - 4 + 5 - 6 - 7 + 8 - 9 + 10 - 11 + 12.$$

Notice that the signs are alternating up to the middle number 6, then they alternate again but in the other order on up to the 12. At this point Huanheng draws lines connecting the numbers on the two ends, so he connects the 1 and the 12, and he connects the −2 and the −11, and so forth. He then points out that this gives us "6 [pairs] of numbers that [add up to] 0." So without actually adding up the numbers in the above list he knows the sum is 0 and explains why. It's a nice idea. Do you see it?

To make it easier to see, I'll rearrange the numbers so that the number pairs are arranged in columns:

$$\begin{array}{cccccc} 1 & -2 & +3 & -4 & +5 & -6 \\ 12 & -11 & +10 & -9 & +8 & -7 \end{array}$$

Now you can see what he had in mind, namely that each pair (i.e. column) adds up to either 13 or −13. There are six pairs altogether and their sum is:

$$13 - 13 + 13 - 13 + 13 - 13 = 0.$$

Notice that there are 6 numbers in this sum since $12/2 = 6$ and notice also that 6 is even, which is what Huanheng told us right upfront.

Though Huanheng doesn't go on to explain how this solves our problem, I think we can guess what he had in mind. Let's try the same idea for the numbers 1 to 100. Notice that $100/2 = 50$ and 50 is even. As before, we write down the numbers 1 to 100 with alternating signs up to the half-way point, i.e. up to 50, and then under these we write the numbers 51 to 100 backwards with alternating signs:

$$\begin{array}{ccccccccc} 1 & -2 & +3 & -4 & +5 & -6 & \cdots & +49 & -50 \\ 100 & -99 & +98 & -97 & +96 & -95 & \cdots & +52 & -51 \end{array}$$

Now each of the 50 columns adds up ±101 with alternating signs, and since 50 is even the total sum must be 0.

So we see that Huanheng's method gives us a beautiful and simple way of arranging the ± signs in front of the numbers 1 to 100 so that the sum is 0. And the same method works for any list of numbers from 1 to $n$ so long as $n$ is even and $n/2$ is also even.

These are nice ideas and I'm happy to congratulate Huanheng for thinking of them. That's why he is the "*Star of Week #5*!"

But this doesn't solve all of our problem. What happens when $n/2$ is *not* even. For example what happens when $n = 50$. In that case $n/2 = 25$ is *odd*, not even. It's easy to see that Huanheng's method doesn't give us a way of arranging the ± signs to make the sum come out to be 0. But maybe there's another method of arranging the signs so that the sum comes out to be 0. Many people said it's not possible, but so far as I could tell, no one was able to give a convincing reason to explain why not. So this is still an open question for us. But here's a hint.

**Hint:** Sometimes the secret to finding the answer to a question is to ask the *right* question. Most of you said that there's no way to get 0 from the numbers 1 to 50. But then that begs the following question:

> *If we can't get* 0*, then what numbers* **can** *we get?*

In fact, it's fun to look at what numbers people were able to get. On the first page of his notes, Huanheng pointed out that if you put minus signs in front of all the *odd* numbers between 1 and 50 and add up, you get $-1-3-5-7\cdots = -625$ and if you put plus signs in front of the *even* numbers between 1 and 50 and add *them* up, you get $2+4+6+8+\cdots = 650$. Then he writes 25, which is the sum of 650 and $-625$, so Huanheng knew that you can get 25 from the numbers 1 to 50. I looked at other students' solutions to see what else people found. Katherine Santos of the Boston Latin Academy (BLA) found a way to get 43. Also Jason Gashi of BLA found a way to get 75, and Kamari St. Paul of BLA found a clever way of getting 51. Of course we can also get 1275, since that's just the sum of the numbers from 1 to 50, as most of you noticed. By changing all the signs, we can also get the negatives of these numbers. Gathering all of this information together we see that the numbers

$$\pm 1275,\ \pm 25,\ \pm 43,\ \pm 75,\ \text{and}\ \pm 51$$

can all be gotten by adding and subtracting the numbers 1 to 50. What do these numbers have in common? If we can get *these* numbers, what *other* numbers do you think we can get? And what numbers *can't* we get? Can we get 0? *Why not?*

Glenn Stevens (Boston University)

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-1 . . . . +49 -50 = 0

-2 +3 . +47 - 48 = 0

-625

-1 -3 -5 -7

+2 +4 +6 +8

650

650

-650

-4 +5 . +45 - 46 = 0

25 ? = 0

#51

51 x 25 + 25

1300

1 - 50

If

1 (everything equal 0 here) +49 - 50 = 0

then

-2 +3 (everything equal 0) +47 ~~#~~ -48
= 0

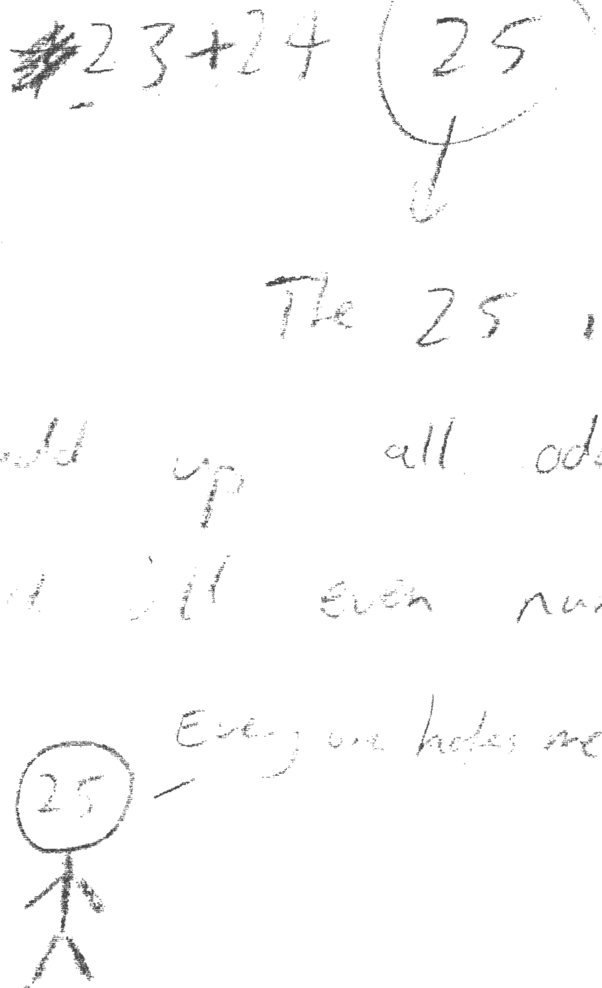
If we keep going

-23+24 (25) +26-27 ≠ 0

The 25 is in the middle.

If we add up all odd numbers (625

and subtract all even numbers -(650




25 is the difference

It's 25 again

1-2-3+4=0 works because there

is 4 numbers $4/2 = 2$ and 2 is a even

If we try 1-10

1-2+3-4+(5 6)-7+8-9+10

This is equal 0

without 5 and 6

~~If~~

If we try 1-12 $12/2 = 6$ 6=even

1-2+3-4+5-6-7+8-9+10-11+12

6sets of numbers that get 0.